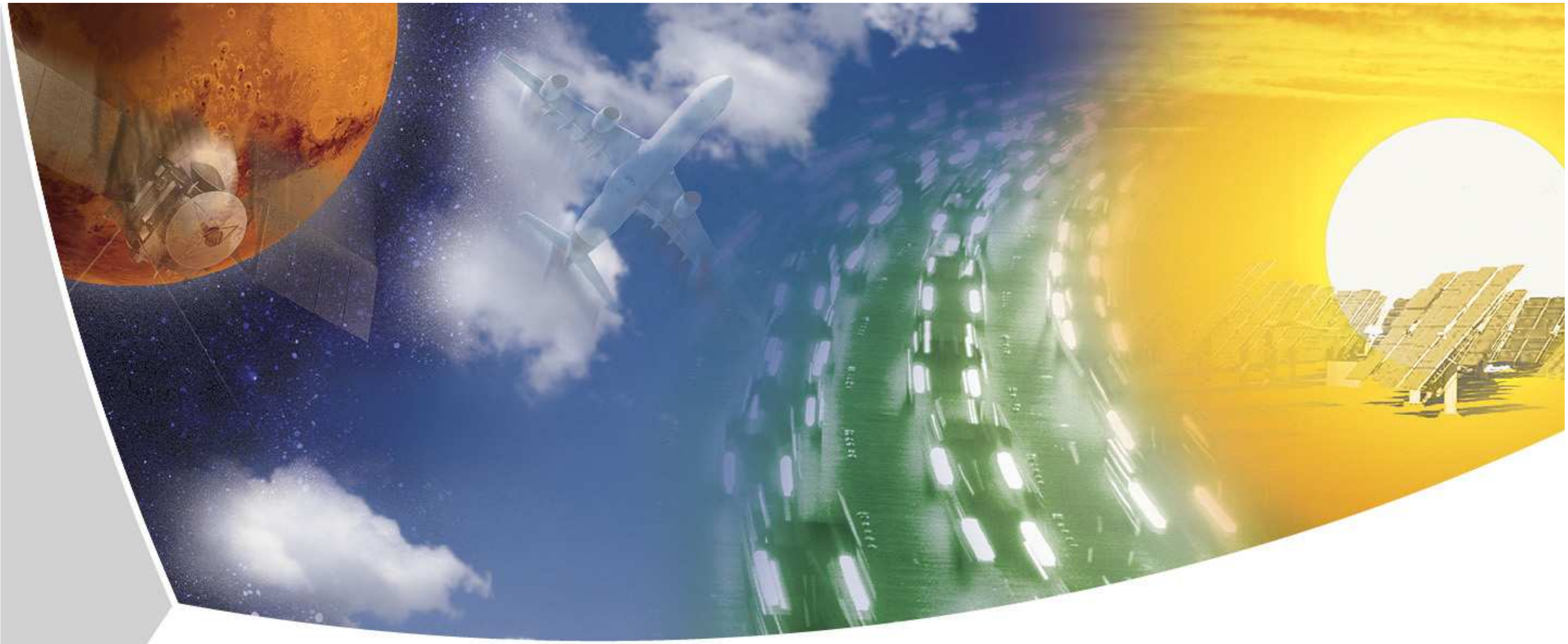

RANS Turbulence Modelling
TAU Training

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Overview

- Introduction
- Averaging
- RANS equations
- Turbulence equations
- RANS turbulence models
- Hints for application

Overview

- **Introduction**
- Averaging
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Introduction

Motivation of turbulence modeling

Aircraft aerodynamics ➔ high Reynolds numbers ➔ flow is turbulent

- Irregular (statistical) fluctuations of all quantities
  - in time ➔ unsteady
  - in space ➔ three-dimensional
- Resolution of all scales for technical applications not feasible

⇨ **Modeling required**

Idea

- Averaging of flow equations
  (Reynolds Averaged Navier-Stokes equations)
- Modeling of influence of turbulence on mean flow

⇨ **RANS turbulence models**

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Averaging

Averaging techniques

Ensemble averaging

$$\overline{\phi}_E(\vec{x},t) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi_n(\vec{x},t)$$

Spatial averaging

$$\overline{\phi}_V(t) = \lim_{\Delta V\to\infty} \frac{1}{\Delta V} \iiint_{\Delta V} \phi(\vec{x},t)\, dV$$

Temporal averaging

$$\overline{\phi}_t(\vec{x}) = \lim_{\Delta t\to\infty} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} \phi(\vec{x},t)\, dt$$

Ergodic hypothesis:
All averages are equal (in steady, homogenoeus turbulence)
➔ Temporal averages usually assumed

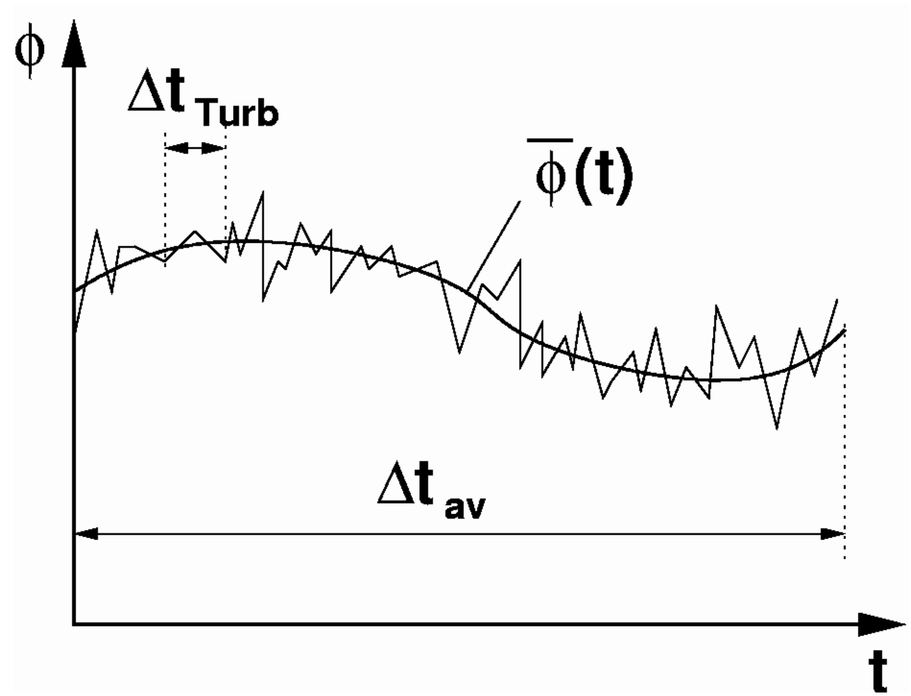
Averaging

Unsteady RANS

- Macroscopically unsteady flow
- Temporal averaging possible, if

$$\Delta t_{Turb} << \Delta t << \Delta t_{av}$$

Spectral gap



Note

- Existence of spectral gap is not guaranteed
- Applicability of URANS is still debated

Averaging

Averages in RANS equations

- **Simple average (Reynolds average)**

$$\phi = \overline{\phi} + \phi' \qquad \textbf{where} \qquad \overline{\phi} \quad \textbf{simple average}$$

$$\phi' \quad \textbf{fluctuation}$$

- **Mass weighted average (Favre average)**

$$\phi = \widetilde{\phi} + \phi'' \qquad \textbf{where} \qquad \widetilde{\phi} = \frac{\overline{\rho\phi}}{\overline{\rho}} \quad \textbf{mass weighted average}$$

$$\phi'' \quad \textbf{fluctuation}$$

Note

- **Mass weighted averages simplify the notation, not the physics**
- **For constant density both mass weighted and simple averages are equal**

Averaging

Averaging rules (1)

- **Averages of fluctuations**

$$\overline{\phi'} = \widetilde{\phi''} = 0$$

- **Averages of averages**

$$\overline{\overline{\phi}} = \overline{\phi} \qquad \widetilde{\widetilde{\phi}} = \widetilde{\phi}$$

- **Averages of differentials**

$$\overline{\left(\frac{\partial \phi}{\partial t}\right)} = \frac{\partial \overline{\phi}}{\partial t}$$

(spectral gap)

$$\overline{\left(\frac{\partial \phi}{\partial x_i}\right)} = \frac{\partial \overline{\phi}}{\partial x_i}$$

Mass weighted differentials accordingly

Averaging

Averaging rules (2)

- **Averages of products**

$$\overline{\phi_1\phi_2} = \overline{\phi_1}\,\overline{\phi_2} + \overline{\phi_1'\phi_2'}$$

Mass weighted average of products accordingly

- **Averages of products with the density**

$$\overline{\rho\phi} = \overline{\rho}\,\widetilde{\phi}$$

$$\overline{\rho\phi''} = 0$$

- **Averages of triple products with the density**

$$\overline{\rho\phi_1\phi_2} = \overline{\rho}\,\widetilde{\phi_1}\widetilde{\phi_2} + \overline{\rho\phi_1''\phi_2''}$$

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RANS equations

Method of derivation

1. Take formal average of exact equation
2. Apply averaging rules
3. Make simplifying assumptions

Example: Momentum equation

1.

$$\overline{\frac{\partial \left(\rho U_i\right)}{\partial t}} + \overline{\frac{\partial}{\partial x_k}\left(\rho U_i U_k\right)} = -\overline{\frac{\partial p}{\partial x_i}} + \overline{\frac{\partial \tau_{ik}}{\partial x_k}}$$

2a.

$$\frac{\partial \left(\overline{\rho U_i}\right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\overline{\rho U_i U_k}\right) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{ik}}{\partial x_k}$$

2b.

$$\frac{\partial \left(\bar{\rho} \tilde{U}_i\right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{\rho}\, \tilde{U}_i \tilde{U}_k\right) + \frac{\partial}{\partial x_k}\left(\bar{\rho} \tilde{R}_{ik}\right) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{ik}}{\partial x_k}$$

where $\bar{\rho} \tilde{R}_{ik} = \overline{\rho u_i'' u_k''}$ **Reynolds stress tensor**

RANS equations

Transport equations

Mass

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{\rho}\, \tilde{U}_k \right) = 0$$

Momentum

$$\frac{\partial \left(\bar{\rho} \tilde{U}_i \right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{\rho}\, \tilde{U}_i \tilde{U}_k \right) + \frac{\partial}{\partial x_k}\left(\bar{\rho} \tilde{R}_{ik} \right) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{ik}}{\partial x_k}$$

Total energy

$$\frac{\partial \left(\bar{\rho} \tilde{E} \right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{\rho}\, \tilde{H} \tilde{U}_k \right) + \frac{\partial}{\partial x_k}\left(\bar{\rho} \tilde{R}_{ik} \tilde{U}_i \right) = \frac{\partial \left(\bar{\tau}_{ik} \tilde{U}_i \right)}{\partial x_k} - \frac{\partial \bar{q}_k}{\partial x_k} + \bar{\rho} D^{(\tilde{k})} - \frac{\partial q_k^{(t)}}{\partial x_k}$$

where

$$\bar{\rho} \tilde{R}_{ik} = \overline{\rho u_i'' u_k''}$$

Reynolds stress tensor

$$\bar{\rho} D^{(\tilde{k})} = \frac{\partial}{\partial x_m}\left(\frac{1}{2} \overline{\rho u_i'' u_i'' u_m''} - \overline{\tau_{im} u_i''} \right)$$

Diffusion flux of turbulent kinetic energy

$$q_k^{(t)} = \overline{\rho h'' u_k''}$$

Turbulent heat flux

Terms require modelling

RANS equations

Thermal equation of state (ideal gas)

$$\bar{p} = \bar{\rho} R \widetilde{T}$$ **where** $R = C_p - C_v$ **specific gas constant**

Caloric equations of state (perfect gas)

- **Specific total energy**

$$\widetilde{E} = \widetilde{e} + \frac{\widetilde{U}_k \widetilde{U}_k}{2} + \widetilde{k}$$

- **Specific total enthalpy**

$$\widetilde{H} = \widetilde{h} + \frac{\widetilde{U}_k \widetilde{U}_k}{2} + \widetilde{k}$$

- **Specific internal energy**

$$\widetilde{e} = C_v \widetilde{T}$$

- **Specific enthalpy**

$$\widetilde{h} = C_p \widetilde{T}$$

where $C_p, C_v = const.$

Specific heats

- **Specific turbulent kinetic energy**

$$\widetilde{k} = \frac{1}{2} \frac{\overline{\rho u_i'' u_i''}}{\bar{\rho}} = \frac{1}{2} \widetilde{R}_{ii}$$

RANS equations

Viscous stress tensor (Newtonian fluid)

$$\bar{\tau}_{ik} = 2\overline{\mu S^*_{ik}} \approx 2\bar{\mu}\tilde{S}^*_{ik}$$

where $\bar{\mu}$ **Averaged molecular viscosity**

$$\tilde{S}^*_{ik} = \frac{1}{2}\left(\frac{\partial \tilde{U}_i}{\partial x_k} + \frac{\partial \tilde{U}_k}{\partial x_i}\right) - \frac{1}{3}\frac{\partial \tilde{U}_m}{\partial x_m}\delta_{ik}$$

Averaged traceless strain rate tensor

Heat flux vector (Fourier's law)

$$\bar{q}_k = -\overline{\lambda \frac{\partial T}{\partial x_k}} \approx -\bar{\lambda}\frac{\partial \tilde{T}}{\partial x_k}$$

where $\bar{\lambda}$ **averaged heat conductivity**

RANS equations

Molecular viscosity (Sutherland's law)

$$\bar{\mu} = \overline{\mu_{ref}\left(\frac{T}{T_{ref}}\right)^{3/2}\frac{T_{ref}+S}{T+S}} \approx \mu_{ref}\left(\frac{\tilde{T}}{T_{ref}}\right)^{3/2}\frac{T_{ref}+S}{\tilde{T}+S}$$

where $S = 110.4K$ **Sutherland's constant**

Thermal conductivity (from Prandtl number)

$$\bar{\lambda} = \frac{\bar{\mu} C_p}{\Pr}$$

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Turbulence equations

Reynolds stress transport equation

1. Write exact momentum equation as

$$N(U_i) = \frac{\partial\left(\rho U_i\right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho U_i U_k\right) + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ik}}{\partial x_k} = 0$$

2. Take following average

$$u_j'' N\left(U_i\right) = 0 \quad \longrightarrow \quad u_j'' N\left(U_i\right) + u_i'' N\left(U_j\right) = 0 \quad \longrightarrow \quad \overline{u_j'' N(U_i) + u_i'' N(U_j)} = 0$$

3. Re-arrange terms by splitting quantities into averages and fluctuations (note that the re-arrangement is not unique)

4. Result

$$\frac{\partial\left(\bar{\rho}\tilde{R}_{ij}\right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{R}_{ij}\tilde{U}_k\right) = \bar{\rho}P_{ij} + \bar{\rho}\Phi_{ij} - \bar{\rho}\varepsilon_{ij} + \bar{\rho}D_{ij} + \bar{\rho}M_{ij}$$

Reynolds stress (transport) equation

Turbulence equations

Terms of the Reynolds stress equation (1)

- **Production**

$$\bar{\rho} P_{ij} = -\bar{\rho}\widetilde{R}_{ik}\frac{\partial \widetilde{U}_j}{\partial x_k} - \bar{\rho}\widetilde{R}_{jk}\frac{\partial \widetilde{U}_i}{\partial x_k}$$

- **Production by gradients of the mean velocity field**
- **Reynolds stresses and mean velocity are provided**
  - ➔ **exact, no modelling needed**

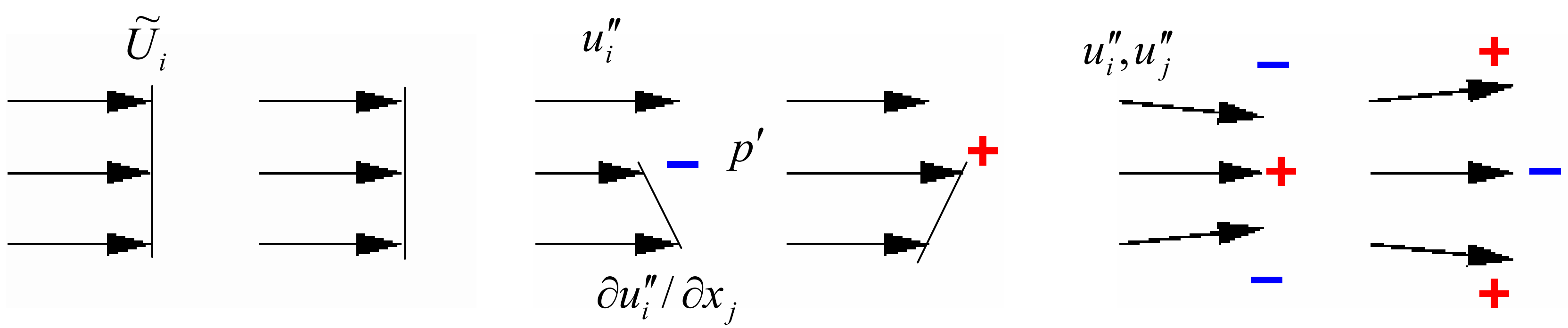
Turbulence equations

Terms of the Reynolds stress equation (2)

- **Re-distribution (pressure-strain correlation)**

$$\bar{\rho}\Phi_{ij} = \overline{p'\left(\frac{\partial u_i''}{\partial x_j} + \frac{\partial u_j''}{\partial x_i}\right)}$$

  - **Excitation of fluctuation u_i'' by fluctuation u_j'' (and vice versa)**



  - **Traceless for constant density ➔ no contribution to energy**
  - **Trace is usually neglected in compressible flow**
  - **Term tends to isotropy in homogeneous mean flow**

Turbulence equations

Terms of the Reynolds stress equation (3)

- **Dissipation**

$$\bar{\rho}\varepsilon_{ij} = \bar{\rho}\varepsilon_{ij}^{(D)} + \frac{2}{3}\bar{\rho}\varepsilon^{(tot)}\delta_{ij}$$

where $$\bar{\rho}\varepsilon_{ij}^{(D)} = \overline{\tau'_{ik}\frac{\partial u''_j}{\partial x_k}} + \overline{\tau'_{jk}\frac{\partial u''_i}{\partial x_k}} - \frac{2}{3}\bar{\rho}\varepsilon^{(tot)}\delta_{ij}$$ **deviatoric (traceless)**

$$\bar{\rho}\varepsilon^{(tot)} = \overline{\tau'_{kl}\frac{\partial u''_k}{\partial x_l}}$$ **total dissipation rate**

- **Dissipation by viscous stresses**
- **Isotropic at smallest scales**
- **Modelled by equation for a length scale determining variable**

Turbulence equations

Terms of the Reynolds stress equation (4)

- **Diffusion**

$$\bar{\rho} D_{ij} = \bar{\rho} T_{ij} + \bar{\rho} D_{ij}^{(v)} + \bar{\rho} D_{ij}^{(p)}$$

  - **Turbulent transport**

$$\bar{\rho} T_{ij} = -\frac{\partial}{\partial x_k}\left[\overline{\rho u_i'' u_j'' u_k''}\right]$$

  - **Viscous diffusion**

$$\bar{\rho} D_{ij}^{(v)} = \frac{\partial}{\partial x_k}\left[\overline{\tau_{ik}' u_j'' + \tau_{jk}' u_i''}\right]$$

  - **Pressure diffusion**

$$\bar{\rho} D_{ij}^{(p)} = \frac{\partial}{\partial x_k}\left[\overline{p' u_i''}\delta_{jk} + \overline{p' u_j''}\delta_{ik}\right]$$

usually neglected

Note

$$\frac{1}{2}\left[\bar{\rho} T_{ii} + \bar{\rho} D_{ii}^{(v)}\right] = \bar{\rho} D^{(\tilde{k})}$$

Contribution to total energy transport equation

Note

All terms represent surface integrals ➔ fluxes ➔ notion of “diffusion”

Turbulence equations

Terms of the Reynolds stress equation (5)

- **Fluctuating mass flux contribution**

$$\bar{\rho} M_{ij} = \overline{u_i''} \frac{\partial}{\partial x_k} \left| -\bar{p}\delta_{jk} + \bar{\tau}_{jk} \right| + \overline{u_j''} \frac{\partial}{\partial x_k} \left| -\bar{p}\delta_{ik} + \bar{\tau}_{ik} \right|$$

  - **Contribution due to fluctuating density**
  - **Only important at high Mach numbers (> 3…5)**
  - **Usually neglected**

Turbulence equations

Special case: Incompressible fluid with constant parameters

Density and viscosity constant $\rho, \mu = const.$

In this case holds

$$D_{ij}^{(v)} - \varepsilon_{ij} = \nu \frac{\partial^2 \overline{R}_{ij}}{\partial x_k \partial x_k} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} = \hat{D}_{ij}^{(v)} - \hat{\varepsilon}_{ij}$$

where $\hat{D}_{ij}^{(v)} = \nu \frac{\partial^2 \overline{R}_{ij}}{\partial x_k \partial x_k}$ **"Incompressible" viscous diffusion**

$\hat{\varepsilon}_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}$ **"Incompressible" dissipation**

with $\nu = \frac{\mu}{\rho} = const.$ **kinematic viscosity**

Turbulence equations

Transport equation of the turbulent kinetic energy

1. Take trace of the Reynolds stress transport equation

2. Consider that

$$\widetilde{R}_{ii} = 2\widetilde{k}$$

3. Result

$$\frac{\partial\left(\overline{\rho}\widetilde{k}\right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\overline{\rho}\widetilde{k}\widetilde{U}_k\right) = \overline{\rho}P^{(\widetilde{k})} + \overline{\rho}\Phi^{(\widetilde{k})} - \overline{\rho}\varepsilon^{(tot)} + \overline{\rho}D^{(\widetilde{k},exact)} + \overline{\rho}M^{(\widetilde{k})}$$

Transport equation for the turbulent kinetic energy or k-equation

Turbulence equations

Terms of the k-equation (1)

- **Production**

$$\bar{\rho} P^{(\tilde{k})} = -\bar{\rho} \widetilde{R}_{ik} \frac{\partial \widetilde{U}_i}{\partial x_k}$$

  - **Production by gradients of the mean velocity field**
  - **Reynolds stresses no longer provided**
    ➔ need modelling

- **Pressure dilatation**

$$\bar{\rho} \Phi^{(\tilde{k})} = \overline{p' \frac{\partial u_k''}{\partial x_k}}$$

  - **Zero for constant density**
  - **Usually neglected for transonic flow**

Turbulence equations

Terms of the k-equation (2)

- **Dissipation**

$$\bar{\rho}\varepsilon^{(tot)} = \overline{\tau_{kl}\frac{\partial u_k''}{\partial x_l}}$$

Total dissipation rate

  - **Dissipation by viscous stresses**
  - **Modelled by equation for a length scale determining variable**

Turbulence equations

Terms of the k-equation (3)

- **Diffusion**

$$\bar{\rho}D^{(\tilde{k},exact)} = \bar{\rho}T^{(\tilde{k})} + \bar{\rho}D^{(\tilde{k},v)} + \bar{\rho}D^{(\tilde{k},p)}$$

– **Turbulent transport**

$$\bar{\rho}T^{(\tilde{k})} = -\frac{1}{2}\frac{\partial}{\partial x_k}\left|\overline{\rho u_i'' u_i'' u_k''}\right|$$

– **Viscous diffusion**

$$\bar{\rho}D^{(\tilde{k},v)} = \frac{\partial}{\partial x_k}\left|\overline{\tau_{ik}' u_i''}\right|$$

– **Pressure diffusion**

$$\bar{\rho}D^{(\tilde{k},p)} = \frac{\partial}{\partial x_k}\left|\overline{p' u_k''}\right|$$

Note $\bar{\rho}T^{(\tilde{k})} + \bar{\rho}D^{(\tilde{k},v)} = \bar{\rho}D^{(\tilde{k})}$

Contribution to total energy transport equation

usually neglected

Note

All terms represent surface integrals ➔ fluxes ➔ notion of "diffusion"

Turbulence equations

Terms of the k-equation (4)

- **Fluctuating mass flux contribution**

$$\bar{\rho} M^{(\tilde{k})} = \overline{u_i''}\left(-\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{ik}}{\partial x_k}\right)$$

  - **Contribution due to fluctuating density**
  - **Only important at high Mach numbers (> 3…5)**
  - **Usually neglected**

Turbulence equations

Special case: Incompressible fluid with constant parameters

Density and viscosity constant $\rho, \mu = const.$

In this case holds

$$D^{(\bar{k},\nu)} - \varepsilon^{(tot)} = \nu \frac{\partial^2 \bar{k}}{\partial x_k \partial x_k} - \nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} = \hat{D}^{(\bar{k},\nu)} - \varepsilon$$

where $\hat{D}^{(\bar{k},\nu)} = \nu \frac{\partial^2 \bar{k}}{\partial x_k \partial x_k}$ **"Incompressible" viscous k-diffusion**

$\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}}$ **(Isotropic) dissipation rate**

with $\nu = \frac{\mu}{\rho} = const.$ **kinematic viscosity**

Turbulence equations

Transport equation for the isotropic dissipation rate

Consider incompressible fluid with $\rho, \mu = const.$

1. Subtract exact and averaged momentum equations

$$\frac{\partial u_i'}{\partial t} + \overline{U}_k \frac{\partial u_i'}{\partial x_k} = -u_k' \frac{\partial \overline{U}_i}{\partial x_k} - u_k' \frac{\partial u_i'}{\partial x_k} - \frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + \frac{\partial \overline{R}_{ik}}{\partial x_k}$$

momentum equation of the fluctuation

2. Multiply by $2\nu \frac{\partial u_i'}{\partial x_l}$ **and average**

3. Result

$$\frac{\partial \varepsilon}{\partial t} + \overline{U}_k \frac{\partial \varepsilon}{\partial x_k} = P^{(\varepsilon)} - \Phi^{(\varepsilon)} + D^{(\varepsilon)} + \nu \frac{\partial^2 \varepsilon}{\partial x_k \partial x_k}$$

Transport equation for the isotropic dissipation rate or ε-equation

Turbulence equations

Terms of the ε-equation

- **Production of dissipation**

$$P^{(\varepsilon)} = -2\nu\left(\frac{\partial \overline{U}_k}{\partial x_l} \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_l}} + \frac{\partial \overline{U}_i}{\partial x_k} \overline{\frac{\partial u_i'}{\partial x_l}\frac{\partial u_k'}{\partial x_l}} + \overline{\frac{\partial u_i'}{\partial x_l}\frac{\partial u_k'}{\partial x_l}\frac{\partial u_i'}{\partial x_k}} + \overline{u_k'\frac{\partial u_i'}{\partial x_l}} \frac{\partial^2 \overline{U}_i}{\partial x_k \partial x_l} \right)$$

- **Dissipation of dissipation**

$$\Phi^{(\varepsilon)} = 2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_l}\frac{\partial^2 u_i'}{\partial x_k \partial x_l}}$$

- **Turbulent diffusion of dissipation**

$$D^{(\varepsilon)} = -\frac{2\nu}{\rho}\frac{\partial}{\partial x_i}\left(\overline{\frac{\partial u_i'}{\partial x_l}\frac{\partial p'}{\partial x_l}} \right) - \nu \frac{\partial}{\partial x_k}\left(\overline{u_k' \frac{\partial u_i'}{\partial x_l}\frac{\partial u_i'}{\partial x_l}} \right)$$

Huge number of terms

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RANS turbulence models

Boussinesq hypothesis

1. Turbulence increases the drag ➔ increase the viscosity

$\bar{\rho}\widetilde{R}_{ij} = -2\mu^{(t)}\widetilde{S}_{ij}^{*}$ **where** $\widetilde{S}_{ii}^{*} = 0$ **(traceless)**

2. Trace of the Reynolds stress tensor

$$\widetilde{R}_{ii} = 2\widetilde{k}$$

3. Extension necessary

$\bar{\rho}\widetilde{R}_{ij} = -2\mu^{(t)}\widetilde{S}_{ij}^{*} + \frac{2}{3}\bar{\rho}\widetilde{k}\delta_{ij}$ **Boussinesq hypothesis**

where $\mu^{(t)}$ **eddy viscosity**

- **To be provided by the turbulence model**
- **Depends on the flow, not on the fluid**

RANS turbulence models

Boussinesq hypothesis for the turbulent heat flux

1. Turbulence increases the heat transfer ➔ increase the heat conductivity

$$q_k^{(t)} = -\lambda^{(t)} \frac{\partial \widetilde{T}}{\partial x_k}$$

Boussinesq hypothesis for the turbulent heat flux

where $\lambda^{(t)}$ **eddy heat conductivity**

- **To be provided by the turbulence model**
- **Depends on the flow, not on the fluid**

2. Relate eddy heat conductivity to eddy viscosity

$$\lambda^{(t)} = \frac{\mu^{(t)} C_p}{\mathrm{Pr}_t}$$

where Pr_t **turbulent Prandtl number**

- **Assumed to be constant**
- **Standard value: Pr_t = 0.9**

RANS turbulence models

Eddy viscosity and length scale

1. Re-write Boussinesq hypothesis

$$\widetilde{R}_{ij} = -2\nu^{(t)}\widetilde{S}_{ij}^{*} + \frac{2}{3}\widetilde{k}\delta_{ij}$$

where $\nu^{(t)} = \dfrac{\mu^{(t)}}{\bar{\rho}}$ **kinematic eddy viscosity**

2. Dimensional analysis

$$\left|\nu^{(t)}\right| = \frac{\left|Length\right|^{2}}{Time} = \left|Velocity\right| \cdot \left|Length\right|$$

3. Turbulent velocity scale

$$\widetilde{k}^{1/2}$$

RANS turbulence models

Length scale supplying variables

General approach based on dimensional analysis

$\nu^{(t)} \propto k^{1/2} L$ **where** L **Turbulent length scale**

Typical length scale supplying variables

- **(Isotropic) dissipation rate**

$$L = \frac{\tilde{k}^{3/2}}{\varepsilon} \quad \Rightarrow \quad \nu^{(t)} = C_\mu \frac{\tilde{k}^2}{\varepsilon} \quad \Rightarrow \text{ k-}\varepsilon \text{ models}$$

- **“Specific” dissipation rate**

$$L = \frac{\tilde{k}^{1/2}}{\omega} \quad \Rightarrow \quad \nu^{(t)} = \frac{\tilde{k}}{\omega} \quad \Rightarrow \text{ k-}\omega \text{ models}$$

Note: Wilcox uses $L = \tilde{k}^{1/2} / C_\mu \omega$

RANS turbulence models

k-ε model (1)

1. Consider k-equation for incompressible fluid

$$\frac{\partial \overline{k}}{\partial t} + \overline{U}_k \frac{\partial \overline{k}}{\partial x_k} = -\overline{R}_{ik} \frac{\partial \overline{U}_i}{\partial x_k} - \varepsilon + \nu \frac{\partial^2 \overline{k}}{\partial x_k \partial x_k} + D^{(\overline{k})}$$

2. Use Boussinesq hypothesis

$$\overline{R}_{ij} = -2\nu^{(t)} \overline{S}_{ij} + \frac{2}{3} \overline{k} \delta_{ij}$$

3. Assume gradient diffusion

$$D^{(\overline{k})} = T^{(\overline{k})} + D^{(\overline{k},p)} = \frac{\partial}{\partial x_k} \left(\frac{\nu^{(t)}}{\sigma_k} \frac{\partial \overline{k}}{\partial x_k} \right)$$

Result

$$\frac{\partial \overline{k}}{\partial t} + \overline{U}_k \frac{\partial \overline{k}}{\partial x_k} = 2\nu^{(t)} \overline{S}_{ij} \overline{S}_{ij} - \varepsilon + \frac{\partial}{\partial x_k} \left[\left(\nu + \frac{\nu^{(t)}}{\sigma_k} \right) \frac{\partial \overline{k}}{\partial x_k} \right]$$

Modelled k-equation

RANS turbulence models

k-ε model (2)

1. Consider exact ε-equation for incompressible fluid

$$\frac{\partial \varepsilon}{\partial t} + \overline{U}_k \frac{\partial \varepsilon}{\partial x_k} = P^{(\varepsilon)} - \Phi^{(\varepsilon)} + D^{(\varepsilon)} + \nu \frac{\partial^2 \varepsilon}{\partial x_k \partial x_k}$$

2. Model production and destruction proportional to k-equation terms

$$P^{(\varepsilon)} = C_{\varepsilon 1} \frac{\varepsilon}{\overline{k}} P^{(\overline{k})} \qquad \Phi^{(\varepsilon)} = C_{\varepsilon 2} \frac{\varepsilon^2}{\overline{k}}$$

3. Assume gradient diffusion

$$D^{(\varepsilon)} = \frac{\partial}{\partial x_k}\left(\frac{\nu^{(t)}}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_k} \right)$$

Result

$$\frac{\partial \varepsilon}{\partial t} + \overline{U}_k \frac{\partial \varepsilon}{\partial x_k} = C_{\varepsilon 1} \frac{\varepsilon}{\overline{k}} P^{(\overline{k})} - C_{\varepsilon 2} \frac{\varepsilon^2}{\overline{k}} + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu^{(t)}}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_k} \right]$$

Modelled ε-equation

RANS turbulence models

k-ε model (3)

Remarks

- **Model needs extensions for near wall region**
  ➔ low (turbulent) Re extensions
- **Boundary condition on ε at walls difficult to specify**
  ➔ Unsatisfying representation of the near-wall region (viscous sub-layer)
- **Results tend to show too high friction**
- **Numerical difficulties observed with compressible solvers**

➔ No k-ε model implemented in TAU

RANS turbulence models

Wilcox k-ω model (1988) (1)

Ideas:

- ε-equation theoretically justified
  - Huge number of complex terms in exact ε-equation
  - Small number of simple terms in modelled ε-equation

    ➔ Deficiencies of k-ε models
- Seek for alternative length scale supplying variable
- Model its transport empirically

  ➔ Overcome k-ε deficiencies

RANS turbulence models

Wilcox k-ω model (1988) (2)

Transport equations (compressible form)

k-equation

$$\frac{\partial \left(\bar{\rho}\tilde{k}\right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{k}\,\bar{U}_k\right) = \bar{\rho}P^{(\tilde{k})} - \bar{\rho}\varepsilon + \bar{\rho}D^{(\tilde{k})}$$

ω-equation

$$\frac{\partial \left(\bar{\rho}\omega\right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{\rho}\omega\bar{U}_k\right) = \bar{\rho}P^{(\omega)} - \bar{\rho}\Phi^{(\omega)} + \bar{\rho}D^{(\omega)}$$

Eddy viscosity

$$\mu^{(t)} = \frac{\bar{\rho}\tilde{k}}{\omega}$$

Relation between ε and ω

$$\mu^{(t)} = C_\mu \frac{\bar{\rho}\tilde{k}^2}{\varepsilon} \quad \Rightarrow \quad \varepsilon = C_\mu \tilde{k}\omega$$

Wilcox: $C_\mu \to \beta^*$

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RANS turbulence models

Wilcox k-ω model (1988) (3)

Terms of the k-equation

- **Production: Boussinesq hypothesis**

$$\bar{\rho}P^{(\tilde{k})} = -\bar{\rho}\tilde{R}_{ij}\frac{\partial \tilde{U}_i}{\partial x_j} = \left(2\mu^{(t)}\tilde{S}^{*}_{ij} - \frac{2}{3}\bar{\rho}\tilde{k}\delta_{ij}\right)\frac{\partial \tilde{U}_i}{\partial x_j} = 2\mu^{(t)}\tilde{S}^{*}_{ij}\tilde{S}_{ij} - \frac{2}{3}\bar{\rho}\tilde{k}\frac{\partial \tilde{U}_i}{\partial x_i}$$

where $\tilde{S}^{*}_{ij} = \tilde{S}_{ij} - \frac{1}{3}\tilde{S}_{kk}\delta_{ij}$ **and** $\tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i}\right)$ **Strain rate tensors**

- **Dissipation: Relation between ε and ω**

$$\bar{\rho}\varepsilon = \beta^{*}\bar{\rho}\tilde{k}\omega$$

- **Diffusion: Gradient hypothesis**

$$\bar{\rho}D^{(\tilde{k})} = \frac{\partial}{\partial x_k}\left[\left(\mu + \sigma^{*}\mu^{(t)}\right)\frac{\partial \tilde{k}}{\partial x_k}\right]$$

RANS turbulence models

Wilcox k-ω model (1988) (4)

Terms of the ω-equation

- **Production: Proportional to k-production**

$$\bar{\rho} P^{(\omega)} = \alpha \frac{\omega}{\tilde{k}} \bar{\rho} P^{(\tilde{k})} = \alpha \frac{\omega}{\tilde{k}} \left(2\mu^{(t)} \tilde{S}_{ij}^{*} \tilde{S}_{ij} - \frac{2}{3} \bar{\rho} \tilde{k} \frac{\partial \tilde{U}_i}{\partial x_i} \right)$$

- **Dissipation: Proportional to k-dissipation**

$$\bar{\rho} \Phi^{(\omega)} = \frac{\beta}{\beta^{*}} \frac{\omega}{\tilde{k}} \bar{\rho} \varepsilon = \beta \bar{\rho} \omega^2$$

- **Diffusion: Gradient hypothesis**

$$\bar{\rho} D^{(\omega)} = \frac{\partial}{\partial x_k} \left[\left(\mu + \sigma \mu^{(t)} \right) \frac{\partial \omega}{\partial x_k} \right]$$

RANS turbulence models

Calibration of the Wilcox k-ω model (1988) (1)

Decaying homogeneous isotropic turbulence

- **No gradients in space**

$$\frac{d\tilde{k}}{dt} = -\beta^* \tilde{k}\omega$$

$$\frac{d\omega}{dt} = -\beta\omega^2$$

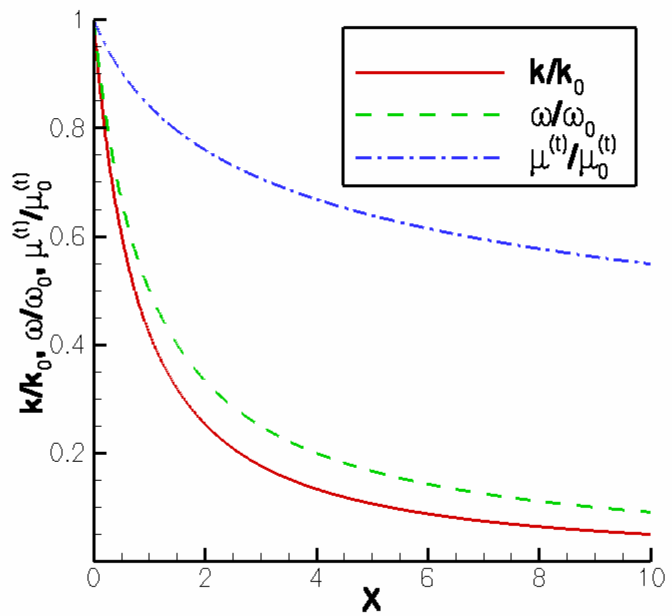
- **Solution**

$$\tilde{k} \propto t^{-\beta^*/\beta}$$

- **Experiments**

$$\tilde{k} \propto t^{-n}$$ **where** $$n = 1.25 \pm 0.06$$

Grid turbulence:
Corresponding spatial decay in parallel flow



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RANS turbulence models

Calibration of the Wilcox k-ω model (1988) (2)

Logarithmic velocity profile on a boundary layer (flat plate)

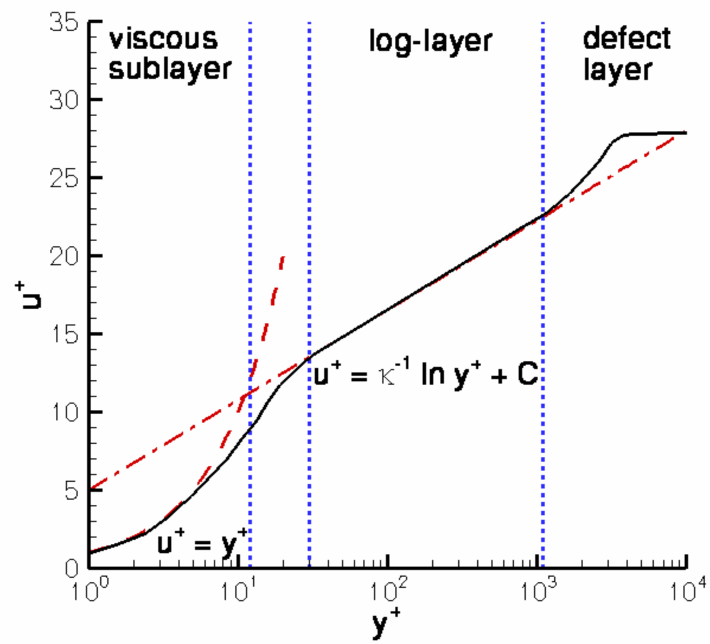
$$\frac{\tilde{U}}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y u_\tau}{\nu}\right) + C$$ **Log-law**

where $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$ **Friction velocity**

$\kappa = 0.41$ **von Karman constant**

$C \approx 5$ **Constant**

Experimentally validated relation



Boundary layer profile (log-plot)

RANS turbulence models

Calibration of the Wilcox k-ω model (1988) (3)

- Assumptions
  - Constant density
  - Zero pressure gradient
  - Log-layer:
    - Viscous stresses negligible
    - Convection negligible
    - Equilibrium: k-production = k-dissipation
      (follows from Bradshaw's relation $R_{xy}/2k = a_1 = \text{const.}$)

- Result for k and ω

$$k = \frac{u_\tau^2}{\sqrt{\beta^*}}$$

$$\omega = \frac{u_\tau}{\sqrt{\beta^*}\kappa y}$$

- Relation for the coefficients

$$\alpha = \frac{\beta}{\beta^*} - \frac{\sigma\kappa^2}{\sqrt{\beta^*}}$$

Calibration of the Wilcox k-ω model (1988) (4)

Choice of coefficients

- **Experiments in equilibrium flow (Bradshaw's relation)**

$$\frac{\overline{R}_{xy}}{\overline{k}} = \frac{u_\tau^2}{\overline{k}} = 0.3 \quad \rightarrow \quad \beta^* = 0.09$$

- **Decaying turbulence**

$$\frac{\beta}{\beta^*} = 1.25 \pm 0.06 \quad \rightarrow \quad \beta = 0.075$$

- **Numerical experiments with varying pressure gradients**

$$\sigma^* = \sigma = 0.5$$

- **Equilibrium boundary layer relation**

$$\alpha = 5/9$$

RANS turbulence models

Comments on the Wilcox k-ω model (1988)

- Standard two-equation model for aerodynamics
- Numerically rather robust
- Rather insensitive to separation
- Foundation of more advanced k-ω type models

RANS turbulence models

Menter baseline (BSL) k-ω model (1994) (1)

Background

- Sensitivity of Wilcox k-ω model to ω-value at boundary layer edge
- k-ε model is insensitive
- Reason: missing cross-diffusion term

Idea

- Use k-ω model near wall and k-ε model further away

Method

- Variable transformation $\varepsilon \rightarrow \omega$ in k-ε model equations
  ➔ Additional cross-diffusion term in ω-equation
- Blending of coefficients from wall (k-ω) to far field (k-ε)

RANS turbulence models

Menter baseline (BSL) k-ω model (1994) (2)

Transport equations

$$\frac{\partial\left(\overline{\rho}\widetilde{k}\right)}{\partial t}+\frac{\partial}{\partial x_k}\left(\overline{\rho}\widetilde{k}\,\overline{U}_k\right)=2\mu^{(t)}\widetilde{S}_{ij}^{*}\widetilde{S}_{ij}-\frac{2}{3}\overline{\rho}\widetilde{k}\frac{\partial\widetilde{U}_k}{\partial x_k}-\beta^{*}\overline{\rho}\widetilde{k}\omega+\frac{\partial}{\partial x_k}\left[\left(\overline{\mu}+\sigma^{*}\mu^{(t)}\right)\frac{\partial\widetilde{k}}{\partial x_k}\right]$$

$$\frac{\partial\left(\overline{\rho}\omega\right)}{\partial t}+\frac{\partial}{\partial x_k}\left(\overline{\rho}\omega\overline{U}_k\right)=\alpha\frac{\omega}{\widetilde{k}}\left(2\mu^{(t)}\widetilde{S}_{ij}^{*}\widetilde{S}_{ij}-\frac{2}{3}\overline{\rho}\widetilde{k}\frac{\partial\widetilde{U}_k}{\partial x_k}\right)-\beta\overline{\rho}\omega^{2}+\frac{\partial}{\partial x_k}\left[\left(\overline{\mu}+\sigma\mu^{(t)}\right)\frac{\partial\omega}{\partial x_k}\right]+C_D$$

Cross-diffusion term

$$C_D=\sigma_d\frac{\overline{\rho}}{\omega}\frac{\partial\widetilde{k}}{\partial x_k}\frac{\partial\omega}{\partial x_k}$$

Eddy viscosity

$$\mu^{(t)}=\frac{\overline{\rho}\widetilde{k}}{\omega}$$

RANS turbulence models

Menter baseline (BSL) k-ω model (1994) (3)

Closure coefficients

$$C = F_1 C^{(k-\omega)} + (1 - F_1) C^{(k-\varepsilon)}$$

where $C = \alpha, \beta, \sigma, \sigma^*, \sigma_d$

Blending function

$$F_1 = \tanh(\Gamma^4)$$

where $\Gamma = \min(\max(\Gamma_1, \Gamma_2), \Gamma_3)$

and $\Gamma_1 = \dfrac{\sqrt{\tilde{k}}}{\beta^* \omega d}$ $\quad \Gamma_2 = \dfrac{500\bar{\mu}}{\bar{\rho}\omega d^2}$ $\quad \Gamma_3 = \dfrac{2\tilde{k}\omega}{\left(\dfrac{\partial \tilde{k}}{\partial x_k}\dfrac{\partial \omega}{\partial x_k}, 10^{-10}\right) d^2}$

d **Wall distance**

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RANS turbulence models

Comments on Menter baseline (BSL) k-ω model (1994)

- **Not very common**
- **Intermediate step towards Menter SST model**
- **Behaviour similar to Wilcox k-ω model expected**

RANS turbulence models

Menter SST model (1994) (1)

Background

- k-ω (and k-ε) models too insensitive to separation
- Reason: Too high eddy viscosity

Idea

- Limit eddy viscosity according to Bradshaw's assumption on shear stress transport (SST)

$$\mu^{(t)} = \frac{\dfrac{\bar{\rho}\tilde{k}}{\omega}}{\max\left(1, \dfrac{F_2 \Omega}{a_1 \omega}\right)}$$

where $\Omega = |\vec{\omega}|$ Vorticity

F_2 Blending function (limit only near walls)

$a_1 = 0.31$ Bradshaw's constant

RANS turbulence models

Menter SST model (1994) (2)

Transport equations

- **Identical with Menter BSL model, except**

- **ω-production**

$$P^{(\omega)} = \alpha \frac{\omega}{\tilde{k}} P^{(\tilde{k})} \quad \rightarrow \quad P^{(\omega)} = \alpha \frac{\bar{\rho}}{\mu^{(t)}} P^{(\tilde{k})}$$

Equivalent, if no SST limitation

- **value of near-wall k-diffusion coefficient**

RANS turbulence models

Comments on Menter SST model (1994)

- Very popular model (recommended)
- More sensitive to separation than Wilcox k-ω
  ➔ Convergence problems possible in case of separation
- Experience
  - Often improved shock position
  - Often improved separation prediction
  - Sometimes too sensitive to separation (e.g. on fine grids)
- Variant: Menter SST model (2003)
  - Replace vorticity in SST limitation by strain rate
  - May reduce sensitivity to separation (experience to be gained)

RANS turbulence models

Further k-ω models implemented in TAU

- LEA
  - Like Wilcox k-ω with deviating eddy viscosity definition
  - Recommended for transonic flows (shock position)
  - Fairly sensitive to separation
- TNT
  - Similar to Wilcox k-ω
  - Includes cross-diffusion term
- Wallin&Johansson (2000) with various ω-equations
  - Explicit Algebraic Reynolds Stress Models (EARSM)
  - Extension of Boussinesq hypothesis by non-linear terms
  - In general similar to LEA
- Non-linear models
  - Similar to EARSM

RANS turbulence models

Far field boundary conditions for k-ω models

k from turbulence intensity

$$\tilde{k}_\infty = \frac{3}{2} U_\infty^2 \left| Tu_\infty \right|^2$$

where Tu_∞ **User defined turbulence intensity**

Note

- **k will rapidly decay along gradient free velocity field (according to isotropic homogeneous turbulence)**

ω from fictitious eddy viscosity

$$\omega_\infty = \frac{\bar{\rho}_\infty \tilde{k}_\infty}{\bar{\mu}_\infty \left(\frac{\mu^{(t)}}{\bar{\mu}} \right)_\infty}$$

where $\left(\frac{\mu^{(t)}}{\bar{\mu}} \right)_\infty < 1$ **User defined**

RANS turbulence models

Wall boundary conditions for k-ω models (1)

k from no-slip condition ➔ no fluctuation

$$\tilde{k}_w = 0$$

ω from viscous sub-layer approximation
➔ requires sub-layer resolution of the grid

$$y_1^+ = \frac{y_1 u_\tau}{\bar{\mu}/\bar{\rho}} \approx 1$$ **where** y_1 **distance of wall-nearest point**

Wall-normal balance between diffusion and dissipation
➔ analytical solution

$$\omega = \frac{6\bar{\mu}}{\beta\bar{\rho}y^2}$$ **where** y **wall-normal coordinate**

Wall value

$$\omega_w = \lim_{y \to 0} \frac{6\bar{\mu}}{\beta\bar{\rho}y^2} \to \infty$$

RANS turbulence models

Wall boundary conditions for k-ω models (2)

Approximations of ω at walls

- **Standard (Menter): Use distance to wall-nearest point**

$$\omega_w = F_w \frac{6\bar{\mu}}{\beta\bar{\rho} y_1^2}$$

where F_w **user defined factor**
Menter: F_w = 10

- **Rudnik: Use reference length of problem**

$$\omega_w = \frac{6\bar{\mu}}{\beta\bar{\rho}\left(10^{-6} L_{ref}\right)^2}$$

where L_{ref} **should be of order one**

- **Wilcox → see wall functions**

Limiting in k-ω models (1)

Realizability

- **k and ω must always remain positive**
- **may be violated during iteration (transient phase)**
- **Remedy: limitation**

Standard limitation (in TAU)

$$\frac{3}{2}\left|\vec{V}\right| Tu_{\min}{}^{2} \le \tilde{k}_{\lim} \le \frac{3}{2}\left|\vec{V}\right| Tu_{\max}{}^{2}$$

where $\left|\vec{V}\right|$ **Local flow velocity**

$Tu_{\min} = 10^{-8}$, $Tu_{\max} = 0.8$ **Local min./max. turb. intensities**

$$\omega_{\lim} \ge \frac{\bar{\rho}\tilde{k}_{\lim}}{\bar{\mu}\left(\frac{\mu^{(t)}}{\bar{\mu}}\right)_{\max}}$$

where $\left(\frac{\mu^{(t)}}{\bar{\mu}}\right)_{\max}$ **User defined (Default: 20000)**

RANS turbulence models

Limiting in k-ω models (2)

Limitation according to Rudnik

$$\tilde{k}_{\lim} \geq C_{\min}^{(\tilde{k})} \tilde{k}_{\infty}$$ **where** $C_{\min}^{(\tilde{k})}, C_{\min}^{(\omega)}$ **User defined (default: 10^{-5})**

$$\omega_{\lim} \geq C_{\min}^{(\omega)} \omega_{\infty}$$

Schwarz limitation (recommended)

$$\tilde{k}_{\lim} \geq C_{\min}^{(\tilde{k})} \tilde{k}_{\infty}$$ **where** $C_{\min}^{(\tilde{k})}$ **User defined (default: 10^{-5})**

$$\omega_{\lim} \geq \omega_{\min}$$

$$\omega_{\min} = \frac{2}{\sqrt{3}} \sqrt{2 \tilde{S}_{ij}^{*} \tilde{S}_{ij}^{*}}$$

Derived from Schwarz inequality

$$\tilde{R}_{\alpha\alpha} \tilde{R}_{\beta\beta} \geq \tilde{R}_{\alpha\beta}^{2}$$

(no summation)

RANS turbulence models

K-production limiting in k-ω models

Equilibrium assumption

- **Calibration requires**

$$P^{(\tilde{k})} \approx \varepsilon$$

- **Will be probably violated during iteration (transient phase)**
- **Violation may cause convergence problems**
- **Remedy: k-production limitation**

Standard technique (Menter)

$P_{\lim}^{(\tilde{k})} \leq C_{\min}^{(P)} \varepsilon$ **where** $C_{\min}^{(P)}$ **User defined (default: 20)**

- **Improves convergence ($C^{(P)}_{min}$ might be reduced to 3…5)**
- **Reduces excessive k-production at stagnation points**

RANS turbulence models

Spalart-Allmaras model (1994) (1)

Ideas

- **Model that is simpler than k-ε**
- **Specifically applicable to aerodynamics**

Approach

- **Define transport equation for (modified) eddy viscosity**
- **Model terms empirically**

$$\frac{\partial\left(\overline{\rho}\tilde{\nu}\right)}{\partial t}+\frac{\partial}{\partial x_k}\left(\overline{\rho}\tilde{\nu}\tilde{U}_k\right)=P^{(\tilde{\nu})}-\Phi^{(\tilde{\nu})}+D^{(\tilde{\nu})}+C_D^{(\tilde{\nu})}$$

- **Eddy viscosity**

$\mu^{(t)}=\overline{\rho}\tilde{\nu}f_{v1}$ **where** $f_{v1}=\dfrac{\chi^3}{\chi^3+c_{v1}^3}$ **Wall-damping function**

$\chi=\overline{\rho}\tilde{\nu}/\overline{\mu}$ **Turbulent Reynolds number**

RANS turbulence models

Spalart-Allmaras model (1994) (2)

Production

$$P^{(\tilde{\nu})} = c_{b1} \tilde{S} \tilde{\nu}$$

where $$\tilde{S} = |\vec{\omega}| + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}$$

and $$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$ **Wall-damping**

$|\vec{\omega}|$ **Vorticity**

d **Wall-distance**

$\kappa = 0.41$ **Von Karman constant**

RANS turbulence models

Spalart-Allmaras model (1994) (3)

Dissipation

$$\Phi^{(\tilde{\nu})} = c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2$$

where $$f_w = g \left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right)^{1/6}$$ **Wall-damping**

and $$g = r + c_{w2} \left(r^6 - r \right)$$

$$r = \frac{\tilde{\nu}}{\tilde{S} \kappa^2 d^2}$$

Note:

No dissipation for large wall-distance d

RANS turbulence models

Spalart-Allmaras model (1994) (4)

Diffusion

$$D^{(\tilde{\nu})} = \frac{\partial}{\partial x_k}\left[\left(\frac{\bar{\mu}+\mu^{(t)}}{\sigma}\right)\frac{\partial\tilde{\nu}}{\partial x_k}\right]$$

Cross-diffusion (non-conservative diffusion, non-linear diffusion)

$$C_D^{(\tilde{\nu})} = c_{b2}\frac{\partial\tilde{\nu}}{\partial x_k}\frac{\partial\tilde{\nu}}{\partial x_k}$$

RANS turbulence models

Spalart-Allmaras model with Edwards modification (1996), SAE

Modification for increased numerical robustness:

- **Use strain rate instead of vorticity**

$$\tilde{S} = \sqrt{2\tilde{S}^*_{ij}\tilde{S}^*_{ij}}$$

- **Re-define auxiliary function r**

$$r = \frac{\tanh\left(\dfrac{\tilde{\nu}}{\kappa^2 d^2 \tilde{S}}\right)}{\tanh|1|}$$

RANS turbulence models

Boundary conditions for Spalart-Allmaras models

Far field:

$$\tilde{\nu}_\infty = \begin{cases} \dfrac{\bar{\mu}_\infty}{\bar{\rho}_\infty}\left(\dfrac{\mu^{(t)}}{\bar{\mu}}\right)_\infty, & d > \delta_{ref} \\ 50\,\dfrac{\bar{\mu}_\infty}{\bar{\rho}_\infty}, & 0 < d \le \delta_{ref} \end{cases}$$

where $\left(\dfrac{\mu^{(t)}}{\bar{\mu}}\right)_\infty$ **User defined (Default: 0.1)**

δ_{ref} **User defined reference boundary layer thickness (Default: 10^{22})**

Smooth walls:

$$\tilde{\nu}_w = 0$$

RANS turbulence models

Comments on Spalart-Allmaras models

- **Very common models**
- **SAE is standard model in TAU**
- **Numerically very robust**
- **Predictions expected similar to Wilcox k-ω**
- **Observations**
  - **Not very sensitive to separation**
  - **In 2D improved high-lift prediction compared to Wilcox k-ω (FLOWer)**

RANS turbulence models

Comments on eddy viscosity models

- Standard for industrial applications
- In general numerically robust and efficient
- Good results for attached flow
- Problems with separated flow

General problems

- Boussinesq hypothesis in general invalid
- Anisotropy of Reynolds stresses near walls not represented
- Excessive dissipation of free vortices (e.g. vortex generators)

RANS turbulence models

Differential Reynolds stress models (DRSM)

Characteristics:
- **Highest level of RANS models**
- **Direct modelling of terms in Reynolds stress transport equation**
  ➔ no Boussinesq hypothesis

Advantages
- **Reynolds stress anisotropy covered**
- **Production term exact**
  ➔ no excessive dissipation of free vortices
- **Probably better suited for separation**

Disadvantages
- **More expensive**
- **Maybe less robust**

RANS turbulence models

Re-distribution modelling (1)

Homogeneous turbulence (no spatial gradients of turbulence)

- **Analytical solution of form**

$$\Phi_{ij} = A_{ij} + M_{ijkl} \frac{\partial \widetilde{U}_k}{\partial x_l}$$

- **Slow term: Return to isotropy (Rotta, 1951)**

$$A_{ij} = -C_1 \frac{\varepsilon}{\widetilde{k}} \left(\widetilde{R}_{ij} - \frac{2}{3} \widetilde{k} \delta_{ij} \right)$$

- **Rapid term: various models**

RANS turbulence models

Re-distribution modelling (2)

Launder-Reece-Rodi model (LRR)

- **Exploit symmetry conditions**

$$M_{ijkl} = a_{ijkl} + a_{jikl}$$

where

$$a_{ijkl} = a_{ljki} = a_{lkji}$$
$$a_{iikl} = 0$$
$$a_{ijjl} = 2\widetilde{R}_{il}$$

Rotta (1951) for constant density

- **Consider most general tensor linear in Reynolds stresses**

- **Result**

$$M_{ijkl}\frac{\partial \widetilde{U}_k}{\partial x_l} = -\hat{\alpha}\left(P_{ij} - \frac{1}{3}P_{kk}\delta_{ij}\right) - \hat{\beta}\left(Q_{ij} - \frac{1}{3}Q_{kk}\delta_{ij}\right) - \hat{\gamma}\widetilde{k}\widetilde{S}^*_{ij}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma} = f(C_2)$

$$P_{ij} = -\widetilde{R}_{ik}\frac{\partial \widetilde{U}_j}{\partial x_k} - \widetilde{R}_{jk}\frac{\partial \widetilde{U}_i}{\partial x_k} \qquad Q_{ij} = -\widetilde{R}_{ik}\frac{\partial \widetilde{U}_k}{\partial x_j} - \widetilde{R}_{jk}\frac{\partial \widetilde{U}_k}{\partial x_i}$$

RANS turbulence models

Re-distribution modelling (3)

Wilcox stress-ω model

Background:
- **LRR model is tied to ε-equation**
- **LRR model needs additional wall-reflection terms (pressure echo effect)**

Idea:
- **Use ω-equation instead of ε-equation**
- **Omit wall reflection terms**

Result:
- **Re-distribution model that works with ω-equation**
- **Applicable to aerodynamics**

RANS turbulence models

Re-distribution modelling (4)

SSG/LRR-ω model

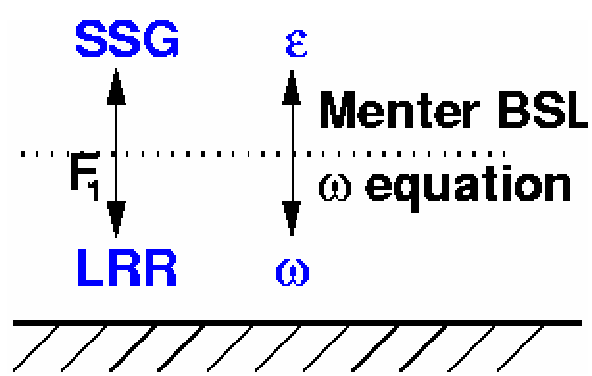
Background:

- **Speziale-Sarkar-Gatski (SSG) model very promising (hom. turbulence)**
- **SSG is non-linear in Reynolds stresses**
- **SSG is tied to ε-equation**

Idea: Combine Wilcox stress-ω (LRR) near walls with SSG in far field
Cast models in identical form → change coefficients only

⇨ **SSG/LRR-ω model**

- **Far field: SSG + ε**
- **Near wall: LRR + ω**
- **Coefficients:**
  Blending function F_1 by Menter
- **BSL-ω-equation by Menter**



RANS turbulence models

Dissipation modelling

Isotropic model (Rotta, 1951)

Background:
- **Dissipation is isotropic on smallest scales**
- **Relation to ε-equation**

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}$$

- **Models based on ω-equation**

$$\varepsilon = \beta^* \tilde{k}\omega \quad \rightarrow \quad \varepsilon_{ij} = \frac{2}{3}\beta^* \tilde{k}\omega\delta_{ij}$$

- **SSG/LRR-ω model: no blend of coefficient**

RANS turbulence models

Diffusion modelling

Simple gradient diffusion hypothesis (SGDH)

- **Diffusion flux proportional gradient of Reynolds stress**

$$D_{ij} = \frac{\partial}{\partial x_k}\left[\left(\bar{\mu} + \sigma^* \mu^{(t,equiv)}\right)\frac{\partial \widetilde{R}_{ij}}{\partial x_k}\right]$$

where $\mu^{(t,equiv)} = \frac{\bar{\rho}\widetilde{k}}{\omega}$ **equivalent eddy viscosity**

Generalized gradient diffusion hypothesis (GGDH) (Daly&Harlow, 1970)

- **Tensorial diffusion coefficient**

$$D_{ij} = \frac{\partial}{\partial x_k}\left[\left(\bar{\mu}\delta_{kl} + \sigma^* \frac{\bar{\rho}\widetilde{R}_{kl}}{\omega}\right)\frac{\partial \widetilde{R}_{ij}}{\partial x_l}\right]$$

Note:

With SSG/LRR-ω model, diffusion coefficients are blended

RANS turbulence models

Modelling of compressibility effects

- Compressibility only important at Ma > 3
- No specific modelling in TAU beyond variable density

$$M_{ij} = 0$$

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RANS turbulence models

Comments on Reynolds stress models in TAU

General

- Only ω-based models available
- Wilcox stress-ω model available with various ω-equations:
    - Wilcox ω-equation
    - Kok ω-equation
    - Menter BSL ω-equation
    - Hellsten ω-equation (not recommended)
- SSG/LRR-ω model requires Menter BSL ω-equation
- Generalized diffusion preferred (sensitivity to separation)

SSG/LRR-ω (recommended)

- Indication of improved predictions for critical flow fields
    - Shock location and shock induced separation
    - Maximum lift
    - Free vortices
- Resources: CPU: + 65%, Memory: less than +100% (comp'd to Menter SST)

Overview

- Introduction
- Averaging
- RANS equations
- Turbulence equations
- RANS turbulence models
- **Hints for application**

Hints for application

Accuracy issues

For enhanced accuracy
- **Use advanced eddy viscosity models, EARSM or RSM**
- **Reduce artificial dissipation**
- **Ensure high grid quality**

Note:
Accurate models should be less dissipative:
- **Reduced robustness**
- **Unsteadiness due to detected separation**
- **Higher sensitivity to grid imperfections**

Hints for application

Robustness issues

For enhanced robustness
- **Use standard eddy viscosity models**
- **Avoid multigrid for turbulence equations**
- **Reduce work on coarse grids in multigrid for RANS equations (v-cycles)**
- **Use implicit scheme (LUSGS)**

Note:
Robust models are probably more dissipative, i.e. less accurate!

Oscillations in coefficients/residuals may indicate separation
- **Use single grid (suppress excitation)**
- **Use more dissipative model (suppress separation) ?**
- **Increase artificial dissipation (suppress separation) ?**